$\boxed{\text{The Bernoulli Function}}$

a useful way to relate velocity + pressure, especially when momentum advection is important.

like uu_x

Recall 1-D SW $\boxed{x \text{ mom}}$

$$\cancel{u_t} + uu_x = -g\eta_x - \cancel{R/h}$$

for steady, frictionless flow

$$\left(\tfrac{1}{2}u^2 + g\eta\right)_x = 0 \quad \text{or} \quad B_x = 0 \Rightarrow \boxed{B = \text{const.}}$$

where $\boxed{B \equiv \tfrac{1}{2}u^2 + g\eta}$ = 1-D SW Bernoulli Function

$\Rightarrow$ η drops $\Longleftrightarrow$ u is faster

Physically: water speeds up when it flows "downhill"

For the 2-D SW equations

steady, frictionless, irrotational $\Rightarrow$ $\zeta = \underbrace{v_x - u_y = 0}_{*}$; $f = 0$

$\boxed{\text{x mom}}$ $\quad uu_x + vu_y + g\eta_x = 0$

$\boxed{\text{y mom}}$ $\quad uv_x + vv_y + g\eta_y = 0$

Rewrite in tricky way:

$$uu_x + \underbrace{vv_x - vv_x}_{=0} + vu_y + g\eta_x = 0$$

$$uv_x + \overbrace{uu_y - uu_y} + vv_y + g\eta_y = 0$$

or

$\boxed{\text{x mom}}$ $\quad \left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right)_x - v\left(\cancel{v_x - u_y}\right)^{0\,(*)} + g\eta_x = 0$

$\boxed{\text{y mom}}$ $\quad \left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right)_y + u\left(\cancel{v_x - u_y}\right)^{0\,(*)} + g\eta_y = 0$

combine into single vector equation

$$\nabla \underbrace{\left(\tfrac{1}{2}\underline{u}^2 + g\eta\right)}_{B\,=\,\text{2D SW Bernoulli function}} = 0$$

$\underline{u} = (u, v)$

$\underline{u}^2 = \underline{u}\cdot\underline{u} = u^2 + v^2$

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$\boxed{\nabla B = 0}$ so $B = \text{const.}$ for all (x, y)

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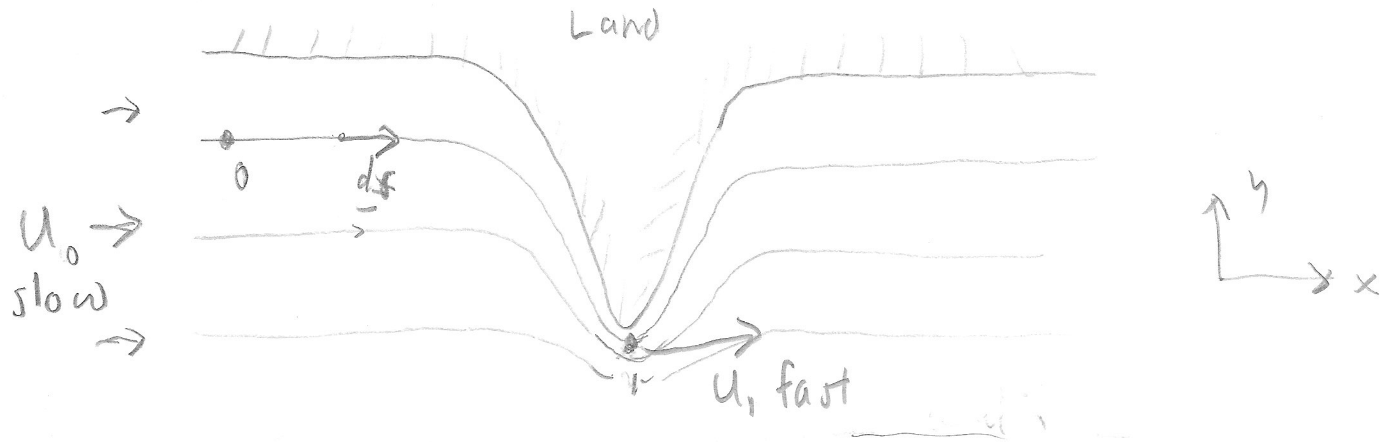
3

(no p.(3))

Application of Bernoulli Fn. to SW flow past a headland.



Flow streamlines - no separation (like potential flow)

Integrating $\nabla B = 0$ along a streamline:

$$\int_{x_0}^{x_1} \nabla B \cdot \underline{ds} = B \Big|_{x_0}^{x_1} = 0$$

$$\Rightarrow \quad \frac{1}{2} u_0^2 + g\eta_0 = \frac{1}{2} u_1^2 + g\eta_1$$

$$\Rightarrow \quad g(\eta_1 - \eta_0) = \frac{1}{2}(u_0^2 - u_1^2)$$

so since the flow is faster at x_2

surface height drops from x_1 to x_2

e.g. if $u_0 = 0.5$ m/s and $u_1 = 1$ m/s

$$\eta_1 - \eta_0 = \Delta\eta = \frac{1}{2g}(.25 - 1)\ \text{m} = \frac{-.75}{20}\ \text{m} \approx -4\ \text{cm}$$

(2)

The Bernoulli Function: 3D

$\underline{u} = (u, v, w)$ $\qquad \hat{k} = (0, 0, 1)$

$\boxed{\underline{x}\text{ mom}}$

$$\underbrace{\frac{D\underline{u}}{Dt}}_{\text{can be written as}} + f\hat{k}\times\underline{u} = -\frac{1}{\rho_0}\nabla p - \frac{g\rho}{\rho_0}\hat{k} + \nabla\cdot(A\nabla\underline{u})$$

$\underline{u}^2 \equiv \underline{u}\cdot\underline{u}$

$$\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = \frac{\partial\underline{u}}{\partial t} + \nabla\left(\frac{1}{2}\underline{u}^2\right) + \underline{\omega}\times\underline{u}$$

where $\underline{\omega} = \nabla\times\underline{u}$ vorticity

Assume
- steady
- inviscid $(A = 0)$
- constant density: $\rho = \rho_0$ along streamlines

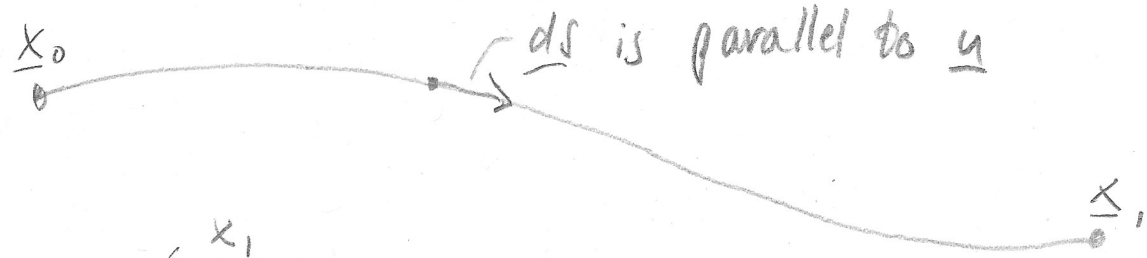
$$\Rightarrow \quad g\hat{k} = \nabla(gz)$$

Leaving

$\boxed{\underline{x}\text{ mom}}$

$$\underbrace{\nabla\left(\frac{1}{2}\underline{u}^2 + \frac{p}{\rho} + gz\right)}_{\text{the Bernoulli Function: 3D}} + (\underline{\omega} + f\hat{k})\times\underline{u} = 0$$

If we integrate along a streamline



then $\displaystyle\int_{\underline{x}_0}^{\underline{x}_1} (\underline{\omega} + f\hat{k}) \times \underline{u} \cdot \underline{ds} = 0$

because $(\;\ast\;) \times \underline{u}$ is $\perp$ to $\underline{u}$

$$\Rightarrow \quad (\ast) \quad \boxed{\frac{1}{2}\underline{u}^2 + \frac{p}{\rho} + gz = B = \text{constant along the streamline}}$$

Bernoulli Function

For SW flow $\quad p = \rho g(\eta - z)$

so $\quad \frac{1}{2}\underline{u}^2 + g\eta \quad = \quad$ constant (our 2-D Bernoulli Fn.)

(for steady, frictionless flow)

So as long as we are integrating along a streamline, B is still constant, even for rotational flow where $\zeta = v_x - u_y \neq 0$ and $f \neq 0$.

And we can analyze more realistic flows using path integrals:

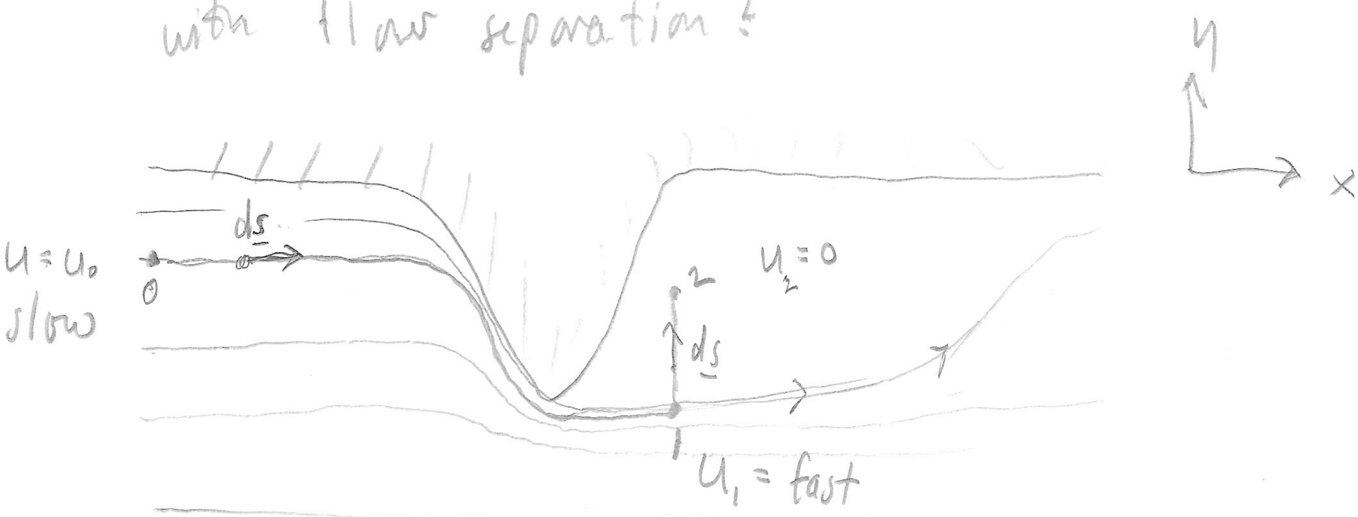
2D SW flow $\boxed{\underset{\sim}{x} \text{ mom}}$

$$\frac{\partial \underline{u}}{\partial t} + \nabla\underbrace{\left(\tfrac{1}{2}\underline{u}^2 + g\eta\right)}_{B} + (\zeta + f)\hat{k} \times \underline{u} = \underline{\text{Friction}}$$

$\zeta = v_x - u_y$ = vertical component of vorticity

You can take path integrals of any of these, along any path.

e.g. What is η in the lee of a headland with flow separation?



We already did $\displaystyle\int_{\underline{x}_0}^{\underline{x}_1} \boxed{\underset{\sim}{x} \text{ mom}} \cdot d\underline{s}$ found $\eta_1 < \eta_0$

$$\int_{\underline{x}_1}^{\underline{x}_2} \boxed{\underline{x}\text{ mom}} \cdot d\underline{s}$$ steady, $f=0$, frictionless

$v=0$ on path, $\zeta = -\frac{\partial u}{\partial y}$

$$\frac{1}{2}\underbrace{\cancel{u_2^2}}_{0} + g\eta_2 - \frac{1}{2}u_1^2 - g\eta_1 + \underbrace{\int_{y_1}^{y_2} \overbrace{-\frac{\partial u}{\partial y}\, u}^{-\left(\frac{1}{2}u^2\right)_y}\, dy}_{\text{from } \int_{\underline{x}_1}^{\underline{x}_2} \zeta \hat{k} \times \underline{u} \cdot d\underline{s}} = 0$$

$\underline{u} = u\hat{i}$

note $\hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \cdot d\underline{s} = dy$ ✓

$$\Rightarrow g\eta_2 - \cancel{\frac{1}{2}u_1^2} - g\eta_1 - \frac{1}{2}\underbrace{\cancel{u_2^2}}_{0} + \cancel{\frac{1}{2}u_1^2} = 0$$

wow!

$$\Rightarrow \boxed{\eta_2 = \eta_1}$$ (even though speed changed $u_1 \to 0$)

★ Bernoulli <u>not</u> conserved across separation region.

Low pressure on lee $\Rightarrow$ form drag.

Similarly, you can allow time-dependence: $\int_{\underline{x}_0}^{\underline{x}_1} \frac{\partial \underline{u}}{\partial t} \cdot d\underline{s}$